Production, Liquidity, and Futures Price Dynamics *

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This paper examines the optimal design of a futures hedge program for the competitive firm under output price uncertainty. All futures contracts are unbiased and marked to market in that they require interim cash settlement of gains and losses. The futures price dynamics follows a first-order autoregression with a random walk serving as a special case. The firm's futures hedge program is constituted of an endogenous provision for premature termination, which depends on how the futures prices are autocorrelated. Succinctly, the firm voluntarily commits to premature liquidation of its futures position on which the interim loss incurred exceeds a predetermined threshold level if the futures prices are positively autocorrelated. In this case, the liquidity constrained firm optimally opts for an over-hedge if its preferences exhibit either constant or increasing absolute risk aversion. If the futures prices are uncorrelated or negatively autocorrelated, the firm prefers to be liquidity unconstrained and thus adopts a full-hedge to completely eliminate the output price risk.

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INTRODUCTION

Firms take liquidity risk seriously when devising their risk management strategies.[1] Failure to do so may lead even technically solvent firms to bankruptcy. An apposite example of this sort is the disaster at Metallgesellschaft A. G. (MG), the fourteenth largest industrial firm in Germany.[2]

In December 1993, MG nearly went bankrupt owing to huge losses incurred by its U.S. subsidiary, MG Refining and Marketing, Inc. (MGRM), in oil futures markets. As a marketing device, MGRM offered long-term contracts for oil and refined oil products that allow its customers to lock in fixed prices up to 10 years into the future. To hedge the oil price risk, MGRM took on large positions in energy derivatives, primarily in oil

[1] According to the Committee on Payment and Settlement Systems (1998), liquidity risk is one of the risks that users of derivatives and other financial contracts must consider.

[2] Another example is the debacle of Long-Term Capital Management (Jorion, 2001).

futures. When oil prices plummeted in 1993, MGRM was unable to meet its variation margin payments due to the denial of credit from its banks.[3] This debacle resulted in a $2.4 billion rescue package coupled with a premature liquidation of its futures positions *en masse* so as to keep MG solvent (Culp and Miller, 1995).

While basis risk in oil futures would certainly imply that MG's hedge did not successfully lock in value (Ross, 1997; Hilliard, 1999; Neuberger, 1999), Mello and Parsons (1995, 2000) identify the funding requirements of MG's hedging strategy as one of the central causes of the problem. Indeed, Mello and Parsons (1995, 2000) show that a perfect hedge does not create its own liquidity, and that the inability to fund a hedging strategy to its end is a serious defect in the design of many popular hedging strategies. In light of these findings, the purpose of this paper is to examine whether there is any role of liquidity constraints in the optimal design of a futures hedge program that allows an endogenously determined provision for terminating the program.

This paper develops a dynamic variant model of the competitive firm under output price uncertainty (Sandmo, 1971). Specifically, the firm produces a single commodity that is sold at the end of the planning horizon. Since the subsequent spot output price is not known ex ante, the firm trades unbiased futures contracts for hedging purposes. All of the unbiased futures contracts are marked to market in that they require cash settlement of gains and losses at the end of each period. The futures price dynamics is assumed to follow a first-order autoregression that includes a random walk as a special case.

To meet the indeterminate interim loss from its futures position, the firm puts aside a fixed amount of capital earmarked for its futures hedge program. The firm commits to premature liquidation of its futures position on which the interim loss incurred exhausts the earmarked capital. The capital commitment as such constitutes a liquidity constraint, where the choice of the former dictates the severity of the latter. The presence of the liquidity constraint is shown to truncate the firm's payoff profile, which plays a pivotal role

[3]Culp and Hanke (1994) report that "four major European banks called in their outstanding loans to MGRM when its problems became public in December 1993. Those loans, which the banks had previously rolled-over each month, denied MGRM much needed cash to finance its variation margin payments and exacerbated its liquidity problems."

in shaping the firm's optimal production and hedging decisions. Given that the futures prices are autocorrelated, the resulting intertemporal linkage is likely to induce the firm to consider a provision for premature termination of its futures hedge program.

In the benchmark case that the firm is not liquidity constrained, the celebrated separation and full-hedging theorems of Danthine (1978), Holthausen (1979), and Feder, Just, and Schmitz (1980) apply. The separation theorem states that the firm's production decision depends neither on its risk attitude nor on the underlying output price uncertainty. The full-hedging theorem states that the firm should completely eliminate its output price risk exposure by adopting a full-hedge via the unbiased futures contracts.

When the choice of the capital commitment, i.e., the severity of the liquidity constraint, is endogenously determined by the firm, it is shown that the firm voluntarily chooses to limit the amount of capital earmarked for its futures hedge program should the futures prices be positively autocorrelated. A positive autocorrelation implies that a loss from a futures position tends to be followed by another loss from the same position. The firm as such finds premature liquidation of its futures position to be ex-post optimal. The severity of the liquidity constraint is thus chosen by the firm to strike a balance between ex-ante and ex-post efficient risk sharing. The liquidity constrained firm is shown to optimally opt for an over-hedge if its preferences exhibit either constant or increasing absolute risk aversion. If the futures prices are uncorrelated or negatively autocorrelated, premature liquidation of the futures position is never ex-post optimal, thereby making the firm prefer not to be liquidity constrained and the separation and full-hedging theorems follow.

In a similar model in which the competitive firm faces an exogenous liquidity constraint and the futures price dynamics follows a random walk, Lien (2003) shows the optimality of an under-hedge. Wong (2004a, 2004b) and Wong and Xu (2006) further show that the liquidity constrained firm optimally cuts down its production. These results are in line with those of Paroush and Wolf (1989) in that the presence of residual unhedgeable risk would adversely affect the hedging and production decisions of the competitive firm under output price uncertainty. In contrast, this paper allows not only an endogenous liquidity constraint but also a first-order autocorrelation of the futures price dynamics. The latter renders the futures prices predictability of which the firm has incentives to take advantage. This explains why an over-hedge, coupled with a commitment to premature liquidation, is optimal when the futures prices are positively autocorrelated. When the futures prices are uncorrelated or negatively autocorrelated, premature liquidation is suboptimal and thus the firm adopts a full-hedge. In the case of an exogenously liquidity constraint, premature liquidation is inevitable so that an under-hedge is called for to limit the potential loss due to a lack of liquidity. The disparate results thus identify factors such as the predictability of futures prices, the severity of liquidity constraints, and the attitude of risk preferences to be crucial for the optimal design of a futures hedge program.

The rest of this paper is organized as follows. The next section delineates a two-period model of the competitive firm facing both output price uncertainty and endogenous liquidity risk. The subsequent section characterizes the solution to the model. The penultimate section constructs a numerical example based on a negative exponential utility function to help understand the findings. The final section concludes.

THE MODEL

Consider a dynamic variant model of the competitive firm under output price uncertainty *à la* Sandmo (1971). There are two periods with three dates, indexed by $t = 0$, 1, and 2. To begin, the firm produces a single commodity according to a deterministic cost function, $c(q)$, where $q \geq 0$ is the output level chosen by the firm at $t = 0$. The cost function, $c(q)$, is assumed to satisfy that $c(0) = c'(0) = 0$, and that $c'(q) > 0$ and $c''(q) > 0$ for all $q > 0$. The firm sells its entire output, q, at $t = 2$ at the then prevailing spot price, $\tilde{p}_2$, that is not known ex ante.[4] Interest rates in both periods, however, are known with certainty at $t = 0$. To simplify notation, the interest factors are henceforth suppressed by compounding all cash flows to their futures values at $t = 2$.

[4]Throughout the paper, random variables have a tilde (~) while their realizations do not.

To hedge its exposure to the price risk, the firm can trade infinitely divisible futures contracts at $t = 0$. Each of the futures contracts calls for delivery of one unit of the commodity at $t = 2$, and is marked to market at $t = 1$. Let f_t be the futures price at date t ($t = 0$, 1, and 2). While the initial futures price, f_0, is predetermined and known at $t = 0$, the other futures prices, $\tilde{f}_1$ and $\tilde{f}_2$, are regarded as random variables ex ante. In the absence of basis risk, the futures price at $t = 2$ must be set equal to the spot price at that time by convergence so that $\tilde{f}_2 = \tilde{p}_2$.

The futures price dynamics is specifically modeled by assuming that $\tilde{f}_t = f_{t-1} + \tilde{\varepsilon}_t$ for $t = 1$ and 2, where $\tilde{\varepsilon}_2 = \rho\tilde{\varepsilon}_1 + \tilde{\delta}$, ρ is a scalar, and $\tilde{\varepsilon}_1$ and $\tilde{\delta}$ are two random variables independent of each other. To focus on the firm's hedging motive, vis-à-vis its speculative motive, $\tilde{\varepsilon}_1$ and $\tilde{\delta}$ are assumed to have means of zero so that the initial futures price, f_0, is unbiased and set equal to the unconditional expected value of the random spot price at $t = 2$, $\tilde{p}_2$. The futures price dynamics as such is a first-order positive or negative autoregression, depending on whether ρ is positive or negative, respectively. If $\rho = 0$, the futures price dynamics becomes a random walk.

The firm's futures hedge program is delineated by a pair, (h, k), where $h \geq 0$ is the number of the futures contracts sold by the firm at $t = 0$, and $k \geq 0$ is a threshold level of interim losses incurred from the firm's futures position, h, due to marking to market at $t = 1$.[5] Specifically, the firm suffers a loss (or enjoys a gain if negative) of $(f_1 - f_0)h$ from its futures position, h, at $t = 1$. The firm commits to prematurely liquidating its futures position, h, at $t = 1$ if $(f_1 - f_0)h > k$, implying that its random profit at $t = 2$ in this case is given by

$$\tilde{\pi}_\ell = \tilde{p}_2 q + (f_0 - f_1)h - c(q) = (f_0 + \rho\varepsilon_1 + \tilde{\delta})q + \varepsilon_1(q - h) - c(q). \tag{1}$$

Otherwise, the firm holds its futures position until $t = 2$ so that its random profit at that time becomes

$$\tilde{\pi}_c = \tilde{p}_2 q + (f_0 - \tilde{f}_2)h - c(q) = f_0 q + (\varepsilon_1 + \rho\varepsilon_1 + \tilde{\delta})(q - h) - c(q). \tag{2}$$

[5] It is shown in the appendix that the firm would never opt for a long futures position, i.e., $h < 0$.

The firm is risk averse and possesses a von Neumann-Morgenstern utility function, $u(\pi)$, defined over its profit at $t = 2$, π, where $u'(\pi) > 0$ and $u''(\pi) < 0$.[6] The firm's decision problem at $t = 0$ is to chooses an output level, q, and devise a futures hedge program, (h, k), so as to maximize the expected utility of its random profit at $t = 2$:

$$\max_{q,h,k} \int_{-\infty}^{k/h} \mathrm{E}_\delta[u(\tilde{\pi}_c)]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 + \int_{k/h}^{\infty} \mathrm{E}_\delta[u(\tilde{\pi}_\ell)]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1, \tag{3}$$

where $\mathrm{E}_\delta(\cdot)$ is the expectation operator with respect to the probability density function of $\tilde{\delta}$, $g(\varepsilon_1)$ is the probability density function of $\tilde{\varepsilon}_1$, and $\tilde{\pi}_\ell$ and $\tilde{\pi}_c$ are defined in Eqs. (1) and (2), respectively. The firm's futures position, h, is said to be an under-hedge, a full-hedge, or an over-hedge if, and only if, h is smaller than, equal to, or greater than q, respectively.

The Kuhn-Tucker conditions for program (3) are given by[7]

$$\int_{-\infty}^{k^*/h^*} \mathrm{E}_\delta\{u'(\tilde{\pi}_c^*)[f_0 + \varepsilon_1 + \rho\varepsilon_1 + \tilde{\delta} - c'(q^*)]\}g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$+ \int_{k^*/h^*}^{\infty} \mathrm{E}_\delta\{u'(\tilde{\pi}_\ell^*)[f_0 + \varepsilon_1 + \rho\varepsilon_1 + \tilde{\delta} - c'(q^*)]\}g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 = 0, \tag{4}$$

$$- \int_{-\infty}^{k^*/h^*} \mathrm{E}_\delta[u'(\tilde{\pi}_c^*)(\varepsilon_1 + \rho\varepsilon_1 + \tilde{\delta})]g(\varepsilon_1)\,\mathrm{d}\varepsilon_1 - \int_{k^*/h^*}^{\infty} \mathrm{E}_\delta[u'(\tilde{\pi}_\ell^*)]\varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1$$

$$-\mathrm{E}_\delta[u(\tilde{\pi}_{c0}^*) - u(\tilde{\pi}_{\ell 0}^*)]g(k^*/h^*)k^*/h^{*2} = 0, \tag{5}$$

and

$$\mathrm{E}_\delta[u(\tilde{\pi}_{c0}^*) - u(\tilde{\pi}_{\ell 0}^*)]g(k^*/h^*)/h^* \geq 0, \tag{6}$$

where $\tilde{\pi}_{\ell 0}^* = f_0 q^* + [(1+\rho)k^*/h^* + \tilde{\delta}]q^* - k^* - c(q^*)$, $\tilde{\pi}_{c0}^* = f_0 q^* + [(1+\rho)k^*/h^* + \tilde{\delta}](q^* - h^*) - c(q^*)$, and an asterisk (*) signifies an optimal level. Should $k^* < \infty$, condition (6) holds with equality.

[6]The risk-averse behavior of the firm can be motivated by managerial risk aversion (Stulz, 1984), corporate taxes (Smith and Stulz, 1985), costs of financial distress (Smith and Stulz, 1985), and capital market imperfections (Froot, Scharfstein, and Stein, 1993; Stulz, 1990). See Tufano (1996) for evidence that managerial risk aversion is a rationale for corporate risk management in the gold mining industry.

[7]The second-order conditions for program (3) are satisfied given risk aversion and the strict convexity of $c(q)$.

SOLUTION TO THE MODEL

Consider first the benchmark case that the firm is not liquidity constrained, which is tantamount to setting $k = \infty$. The first-order conditions for program (3) in this benchmark case are given by

$$\int_{-\infty}^{\infty} \mathrm{E}_\delta\{u'(\tilde{\pi}_c^0)[f_0 + \varepsilon_1 + \rho\varepsilon_1 + \tilde{\delta} - c'(q^0)]\}g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 = 0, \tag{7}$$

and

$$-\int_{-\infty}^{\infty} \mathrm{E}_\delta[u'(\tilde{\pi}_c^0)(\varepsilon_1 + \rho\varepsilon_1 + \tilde{\delta})]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 = 0, \tag{8}$$

where a nought (^{0}) indicates an optimal level.

Adding Eq. (8) to Eq. (7) yields

$$[f_0 - c'(q^0)]\int_{-\infty}^{\infty} \mathrm{E}_\delta[u'(\tilde{\pi}_c^0)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 = 0. \tag{9}$$

Since $u'(\pi) > 0$, Eq. (9) reduces to $c'(q^0) = f_0$. If $h^0 = q^0$, the left-hand side of Eq. (8) becomes

$$-(1+\rho)u'[f_0 q^0 - c(q^0)]\int_{-\infty}^{\infty} \varepsilon_1 g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 = 0, \tag{10}$$

since $\tilde{\varepsilon}_1$ and $\tilde{\delta}$ have means of zero. Inspection of Eqs. (8) and (10) reveals that $h^0 = q^0$ is indeed the optimal futures position. The following proposition is therefore established.

Proposition 1. *Given that the competitive firm is not liquidity constrained, i.e., $k = \infty$, the firm's optimal output level, q^0, solves*

$$c'(q^0) = f_0, \tag{11}$$

and its optimal futures position, h^0, is a full-hedge, i.e., $h^0 = q^0$.

The intuition of Proposition 1 is as follows. If the firm is not liquidity constrained, its random profit at $t = 2$ is given by Eq. (2) only. The firm could have completely eliminated all the price risk had it chosen $h = q$ within its own discretion. Alternatively put, the degree of price risk exposure to be assumed by the firm should be totally unrelated to its production decision. The optimal output level is then chosen to maximize $f_0 q - c(q)$, thereby yielding q^0 that solves Eq. (11). Since the futures contracts are unbiased, they offers actuarially fair "insurance" to the firm. Being risk averse, the firm finds it optimal to opt for full insurance via a full-hedge, i.e., $h^0 = q^0$. These results are simply the well-known separation and full-hedging theorems of Danthine (1978), Holthausen (1979), and Feder, Just, and Schmitz (1980).

Resume now the original case that the liquidity threshold, k, is endogenously determined by the firm at $t = 0$. To facilitate the exposition, fix $h = q = q^0$ in program (3) to yield

$$\max_k \ u[f_0 q^0 - c(q^0)] \int_{-\infty}^{k/q^0} g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 + \int_{k/q^0}^{\infty} \mathrm{E}_\delta\{u[(f_0 + \rho\varepsilon_1 + \tilde{\delta})q^0 - c(q^0)]\} g(\varepsilon_1)\ \mathrm{d}\varepsilon_1. \tag{12}$$

The Kuhn-Tucker condition for program (12) is given by

$$\Big\{u[f_0 q^0 - c(q^0)] - \mathrm{E}_\delta\{u[(f_0 + \tilde{\delta})q^0 + \rho k^0 - c(q^0)]\}\Big\} g(k^0/q^0)/q^0 \geq 0, \tag{13}$$

where k^0 is the optimal liquidity threshold when $h = q = q^0$. Should $k^0 < \infty$, condition (13) holds with equality.

If $\rho \leq 0$, it follows from $u''(\pi) < 0$, $\mathrm{E}_\delta(\tilde{\delta}) = 0$, and Jensen's inequality that $u[f_0 q^0 - c(q^0)] \geq u[f_0 q^0 + \rho k^0 - c(q^0)] > \mathrm{E}_\delta\{u[(f_0 + \tilde{\delta})q^0 + \rho k^0 - c(q^0)]\}$, and thus $k^0 = \infty$ by condition (13). Since Proposition 1 implies that $h^* = q^* = q^0$ if $k^* = \infty$, it must be the case that $h^* = q^* = q^0$ and $k^* = \infty$ if $k^0 = \infty$. On the other hand, if $\rho > 0$, it is evident that $\mathrm{E}_\delta\{u[(f_0 + \tilde{\delta})q^0 + \rho k - c(q^0)]\}$ is increasing in k. When $k = 0$, it follows from $u''(\pi) < 0$, $\mathrm{E}_\delta(\tilde{\delta}) = 0$, and Jensen's inequality that $\mathrm{E}_\delta\{u[(f_0 + \tilde{\delta})q^0 - c(q^0)]\} < u[f_0 q^0 - c(q^0)]$. Also, for k sufficiently large, it must be the case that $\mathrm{E}_\delta\{u[(f_0 + \tilde{\delta})q^0 + \rho k - c(q^0)]\} > u[f_0 q^0 - c(q^0)]$.

Thus, there exists a unique point, $k^0 \in (0, \infty)$, such that condition (13) holds with equality. Suppose that $k^* = \infty$ but $k^0 < \infty$. It then follows from Proposition 1 that $h^* = q^* = q^0$, which would imply that $k^0 = k^* = \infty$, a contradiction to $k^0 < \infty$. The following proposition summarizes these results.

Proposition 2. *Given that the competitive firm optimally devises its futures hedge program, (h^*, k^*), the firm commits to the optimal liquidation threshold, k^*, that is positive and finite (infinite) if the autocorrelation coefficient, ρ, is positive (non-positive).*

The intuition of Proposition 2 is as follows. If the firm chooses $k = \infty$, risk sharing is ex-ante efficient because the firm can completely eliminate all the price risk. However, this is not ex-post efficient, especially when $\rho > 0$. To see this, note that for any given $k < \infty$ the firm prematurely liquidates its futures position at $t = 1$ for all $\varepsilon_1 \in [k/h, \infty)$. Conditional on premature liquidation, the expected value of f_2 is equal to $f_1 + \rho\varepsilon_1$, which is greater (not greater) than f_1 when $\rho > (\leq)\ 0$. Thus, it is ex-post optimal for the firm to liquidate its futures position prematurely to limit further losses if $\rho > 0$. In this case, the firm chooses the optimal threshold level, k^*, to be finite so as to strike a balance between ex-ante and ex-post efficient risk sharing.[8] If $\rho \leq 0$, premature liquidation is never ex-post optimal and thus the firm chooses $k^* = \infty$.

When $\rho \leq 0$, Proposition 2 implies that $k^* = \infty$. It then follows from Proposition 1 that $h^* = q^* = q^0$, thereby invoking the following proposition.

Proposition 3. *Given that the firm optimally devises its futures hedge program, (h^*, k^*), the firm's optimal output level, q^*, equals the benchmark level, q^0, and its optimal futures position, h^*, is a full-hedge, i.e., $h^* = q^*$, if the autocorrelation coefficient, ρ, is non-positive.*

When $\rho > 0$, Proposition 2 implies that the firm voluntarily chooses to be liquidity constrained, i.e., $k^* < \infty$. Hence, in this case, condition (6) holds with equality and

[8] The firm's commitment to premature liquidation of its futures position on which the interim loss incurred exceeds k^* is evidently time-consistent.

the solution, (q^*, h^*, k^*), solves Eqs. (4), (5), and (6) simultaneously. To facilitate the exposition, the firm's ex-ante decision problem is formulated as a two-stage optimization problem with q fixed at q^*. In the first stage, the firm chooses its optimal liquidation threshold, $k(h)$, for a given futures position, h:

$$k(h) = \arg\max_{k>0} \int_{-\infty}^{k/h} \mathrm{E}_\delta[u(\tilde{\pi}_c)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 + \int_{k/h}^{\infty} \mathrm{E}_\delta[u(\tilde{\pi}_\ell)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1, \tag{14}$$

where $\tilde{\pi}_\ell$ and $\tilde{\pi}_c$ are given in Eqs. (1) and (2) with $q = q^*$, respectively. In the second stage, the firm chooses its optimal futures position, h^*, taking the liquidation threshold, $k(h)$, as given by Eq. (14):

$$\max_h\ \Phi(h) = \int_{-\infty}^{k(h)/h} \mathrm{E}_\delta[u(\tilde{\pi}_c)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 + \int_{k(h)/h}^{\infty} \mathrm{E}_\delta[u(\tilde{\pi}_\ell)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1, \tag{15}$$

where $\tilde{\pi}_\ell$ and $\tilde{\pi}_c$ are given in Eqs. (1) and (2) with $q = q^*$ and $k = k(h)$, respectively. The complete solution is thus given by h^* and $k^* = k(h^*)$.

Differentiating $\Phi(h)$ in Eq. (15) with respect to h, using the envelope theorem, and evaluating the resulting derivative at $h = q^*$ yields

$$\Phi'(q^*) = -(1+\rho)u'[f_0 q^* - c(q^*)] \int_{-\infty}^{k(q^*)/q^*} \varepsilon_1 g(\varepsilon_1)\ \mathrm{d}\varepsilon_1$$

$$- \int_{k(q^*)/q^*}^{\infty} \mathrm{E}_\delta\{u'[(f_0 + \rho\varepsilon_1 + \tilde{\delta})q^* - c(q^*)]\}\varepsilon_1 g(\varepsilon_1)\ \mathrm{d}\varepsilon_1, \tag{16}$$

where $k(q^*)$ solves

$$u[f_0 q^* - c(q^*)] = \mathrm{E}_\delta\{u[(f_0 + \tilde{\delta})q^* + \rho k(q^*) - c(q^*)]\}. \tag{17}$$

It is evident from Eq. (17) that $\rho k(q^*)$ is equal to the risk premium of the zero-mean risk, $\tilde{\delta} q^*$, in the usual Arrow-Pratt sense.

Rewrite Eq. (17) as

$$u[f_0 q^* - c(q^*) + m] = \mathrm{E}_\delta\{u[(f_0 + \tilde{\delta})q^* + \rho k(q^*) - c(q^*) + m]\}, \tag{18}$$

where m can be interpreted as endowed wealth that takes on an initial value of zero. Differentiating Eq. (18) with respect to m and evaluating the resulting derivative at $m = 0$ yields

$$\frac{\partial \rho k(q^*)}{\partial m}\bigg|_{m=0} = \frac{u'[f_0 q^* - c(q^*)] - \mathrm{E}_\delta\{u'[(f_0+\tilde{\delta})q^* + \rho k(q^*) - c(q^*)]\}}{\mathrm{E}_\delta\{u'[(f_0+\tilde{\delta})q^* + \rho k(q^*) - c(q^*)]\}}. \tag{19}$$

If $u(\pi)$ satisfies decreasing, constant, or increasing absolute risk aversion (DARA, CARA, or IARA), $\partial \rho k(q^*)/\partial m$ is negative, zero, or positive, respectively. Using the fact that $\tilde{\varepsilon}_1$ has a mean of zero, Eq. (16) can be written as

$$\Phi'(q^*) = \int_{k(q^*)/q^*}^{\infty} \bigg\{(1+\rho)u'[f_0 q^* - c(q^*)]$$
$$-\mathrm{E}_\delta\{u'[(f_0+\rho\varepsilon_1+\tilde{\delta})q^* - c(q^*)]\}\bigg\}\varepsilon_1 g(\varepsilon_1)\,\mathrm{d}\varepsilon_1. \tag{20}$$

If $u(\pi)$ satisfies CARA (IARA), Eq. (19) implies that

$$u'[f_0 q^* - c(q^*)] = (>)\ \mathrm{E}_\delta\{u'[(f_0+\tilde{\delta})q^* + \rho k(q^*) - c(q^*)]\}. \tag{21}$$

Since $\rho > 0$, Eq. (21) and risk aversion imply that $(1+\rho)u'[f_0 q^* - c(q^*)] > \mathrm{E}_\delta\{u'[(f_0 + \tilde{\delta})q^* + \rho k(q^*) - c(q^*)]\} > \mathrm{E}_\delta\{u'[(f_0 + \rho\varepsilon_1 + \tilde{\delta})q^* - c(q^*)]\}$ for all $\varepsilon_1 > k(q^*)/q^*$. It then follows from Eq. (20) that $\Phi'(q^*) > 0$ and thus $h^* > q^*$ if $u(\pi)$ satisfies either CARA or IARA. The following proposition summarizes these results.

Proposition 4. *Given that the competitive firm optimally devises its futures hedge program, (h^*, k^*), the firm's optimal futures position, h^*, is an over-hedge, i.e., $h^* > q^*$, if the autocorrelation coefficient, ρ, is positive and the firm's utility function, $u(\pi)$, satisfies either constant or increasing absolute risk aversion.*

To see the intuition of Proposition 4, refer to Eqs. (1) and (2). If the firm adopts a full-hedge, i.e., $h = q$, its profit at $t = 2$ remains stochastic due to the residual price risk, $(\rho\varepsilon_1 + \tilde{\delta})q$, that arises from the premature closure of its hedge program at $t = 1$. This creates an income effect because the presence of the liquidity risk reduces the attainable expected utility under risk aversion. To attain the former expected utility level (with no risk), the firm has to be compensated with additional income. Taking away this compensation gives rise to the income effect (see Wong, 1997). Under IARA (DARA), the firm becomes less (more) risk averse and thus is willing (unwilling) to take on the liquidity risk. The firm as such shorts more (less) of the futures contracts so as to enlarge (shrink) the interval, $[k/h, \infty)$, over which the premature liquidation of the futures position at $t = 1$ prevails. Since $\rho > 0$, inspection of Eqs. (1) and (2) reveals that the high (low) realizations of the firm's random profit at $t = 2$ occur when the futures position is (is not) prematurely liquidated at $t = 1$. Being risk averse, the firm would like to shift profits from the high-profit states to the low-profit states. This goal can be achieved by shorting more of the futures contracts, i.e., $h > q$, as is evident from Eqs. (1) and (2). Such an over-hedging incentive is reinforced (alleviated) under IARA (DARA). Thus, the firm optimally opts for an over-hedge, i.e., $h^* > q^*$, under either CARA or IARA.

A NUMERICAL EXAMPLE

To gain more insights into the theoretical findings, this section constructs a numerical example to quantity the severity of the endogenous liquidity constraint as inversely gauged by the optimal liquidation threshold, k^*.

Suppose that the firm has a negative exponential utility function: $u(\pi) = -e^{-\gamma\pi}$, where $\gamma > 0$ is the constant Arrow-Pratt measure of absolute risk aversion. For simplicity, the firm produces one unit of output at zero production costs. The underlying random variables, ε_1 and δ, are both normally distributed with means of zero and variances of 0.01. The initial futures price, f_0, is set equal to unity. Table I reports the optimal short futures position, h^*, and the optimal liquidation threshold, k^*, for different values of the risk aversion coefficient, γ, and the autocorrelation coefficient, ρ.

TABLE I
Optimal Futures Positions and Liquidation Thresholds

	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
	h^*	k^*	h^*	k^*	h^*	k^*
$\rho = 0.1$	1.2313	0.0473	1.0738	0.0997	1.0188	0.1485
$\rho = 0.2$	1.5103	0.0185	1.2385	0.0472	1.1349	0.0737
$\rho = 0.5$	2.1670	0.0036	1.6292	0.0121	1.4354	0.0243

Note. This table reports the optimal futures position, h^*, and the optimal liquidation threshold, k^*. The firm has a negative exponential utility function: $u(\pi) = -e^{-\gamma\pi}$, where γ is a positive constant. The firm produces one unit of output at zero production costs. The underlying random variables, ε_1 and δ, are normally distributed with means of zero and variances of 0.01. The initial futures price, f_0, is set equal to unity.

Table I shows that an over-hedge, i.e., $h^* > 1$, is optimal, which is consistent with the findings in Proposition 4. It is also evident from Table I that k^* decreases as either ρ increases or γ decreases. That is, the firm is willing to commit itself to a more aggressive (i.e., severe) liquidity constraint provided that premature liquidation is indeed ex-post profitable or that the firm is less risk averse and thus does not mind to take on excessive risk.

CONCLUSION

This paper has examined the optimal design of a futures hedge program for the competitive firm under output price uncertainty *à la* Sandmo (1971). To hedge its output price risk exposure, the firm trades unbiased futures contracts that are marked to market and require interim cash settlement of gains and losses. The futures price dynamics follows a first-order autoregression that includes a random walk as a special case.

This paper has shown that the firm's futures hedge program contains an optimal provision for premature termination if the futures prices are positively autocorrelated. Furthermore, the liquidity constrained firm optimally opts for an over-hedge if its preferences exhibit either constant or increasing absolute risk aversion. Since a positive autocorrelation implies that a loss from a futures position tends to be followed by another loss from the same position, the firm finds premature liquidation of its futures position to be ex-post optimal. The severity of the liquidity constraint as such is chosen by the firm to strike a balance between ex-ante and ex-post efficient risk sharing. However, if the futures prices are uncorrelated or negatively autocorrelated, premature liquidation of the futures position is never ex-post optimal. In this case, the firm prefers to be liquidity unconstrained and thus adopts a full-hedge to completely eliminate the output price risk.

APPENDIX

The firm's ex-ante decision problem is to choose a futures position, h, so as to maximize the expected utility of its random profit at $t = 2$, EU:

$$\int_{-\infty}^{k/h} \mathrm{E}_\delta[u(\tilde{\pi}_c)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 + \int_{k/h}^{\infty} \mathrm{E}_\delta[u(\tilde{\pi}_\ell)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1, \tag{A.1}$$

if $h > 0$, and

$$\int_{-\infty}^{k/h} \mathrm{E}_\delta[u(\tilde{\pi}_\ell)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1 + \int_{k/h}^{\infty} \mathrm{E}_\delta[u(\tilde{\pi}_c)]g(\varepsilon_1)\ \mathrm{d}\varepsilon_1, \tag{A.2}$$

if $h < 0$, where $\tilde{\pi}_\ell$ and $\tilde{\pi}_c$ are defined in Eqs. (1) and (2), respectively. In order to solve the firm's optimal futures position, h^*, one has to know which equation, Eq. (A.1) or Eq. (A.2), contains the solution.

Consider first the case that $h > 0$. Differentiating EU as defined in Eq. (A.1) with respect to h and evaluating the resulting derivative at $h \to 0^+$ yields

$$\lim_{h\to 0^+} \frac{\partial EU}{\partial h} = -\int_{-\infty}^{\infty} \mathrm{E}_\delta\{u'[(f_0 + \varepsilon_1 + \rho\varepsilon_1 + \delta)q - c(q)](\varepsilon_1 + \rho\varepsilon_1 + \delta)\}g(\varepsilon_1)\ \mathrm{d}\varepsilon_1. \tag{A.3}$$

Since ε_1 and δ has means of zero, the right-hand side of Eq. (A.3) is simply the negative of the covariance between $u'[(f_0 + \varepsilon_1 + \rho\varepsilon_1 + \delta)q - c(q)]$ and $\varepsilon_1 + \rho\varepsilon_1 + \delta$ with respect to the joint probability density function of ε_1 and δ. It then follows from $u''(\pi) < 0$ that $\lim_{h\to 0^+} \partial EU/\partial h > 0$.

Now, consider the case that $h < 0$. Differentiating EU as defined in Eq. (A.2) with respect to h and evaluating the resulting derivative at $h \to 0^-$ yields

$$\lim_{h\to 0^-} \frac{\partial EU}{\partial h} = -\int_{-\infty}^{\infty} \mathrm{E}_\delta\{u'[(f_0 + \varepsilon_1 + \rho\varepsilon_1 + \delta)q - c(q)](\varepsilon_1 + \rho\varepsilon_1 + \delta)\}g(\varepsilon_1)\ \mathrm{d}\varepsilon_1. \quad \text{(A.4)}$$

Inspection of Eqs. (A.3) and (A.4) reveals that $\lim_{h\to 0^+} \partial EU/\partial h = \lim_{h\to 0^-} \partial EU/\partial h > 0$. Since EU as defined in either Eq. (A.1) or Eq. (A.2) is strictly concave, the firm's optimal futures position, h^*, must be a short position, i.e., $h^* > 0$.

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